

## ATTACHMENT I

### DEFINITION OF A TRIANGULAR GRID BASED ON AN ICOSAHEDRON

A triangular grid based on an icosahedron was first introduced in a meteorological model by Sadourny and others (1968) and Williamson (1969). The approach outlined here, especially the code implementation, is based on the work of Baumgardner (1995).

To construct the triangular grid based on an icosahedron, the unit-sphere, i.e. a sphere with radius 1, is divided into 20 spherical triangles of equal size by placing a plane icosahedron into the sphere (Figure 1). The 12 vertices of the icosahedron touch the sphere, one vertex coincides with the north pole (NP), the opposite one with the south pole (SP), for simplicity.

The 12 vertices are connected by great circles to form 20 main spherical triangles. Since each of the 12 vertices is surrounded by five main spherical triangles (Figure 2), the angles between two sides of the main triangles are  $2\pi/5$  or  $72^\circ$ .

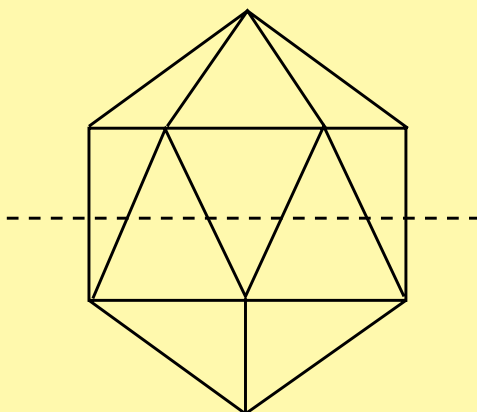


Figure 1. Plane icosahedron consisting of 20 plane triangles.

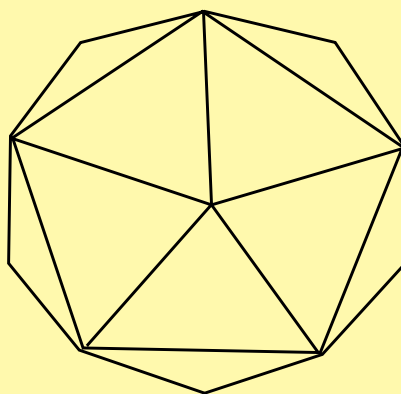


Figure 2. The five main spherical triangles at the north pole.

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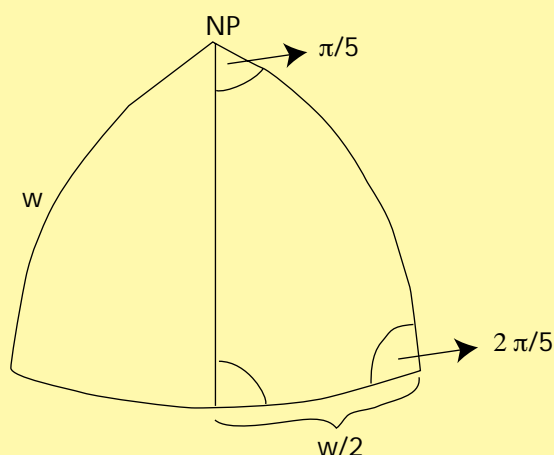


Figure 3. One main spherical triangle at the North Pole.

The length  $w$  of a main triangle side follows from Figure 3 and equation (1):

$$\cos \frac{1}{2} w = \frac{\cos \frac{\pi}{5}}{\sin 2 \frac{\pi}{5}} = \frac{1}{2 \sin \frac{\pi}{5}} \quad (1)$$

Thus  $w \approx 1.107149$ . On the unit-sphere,  $w$  is identical to  $\pi/2$  minus  $\varphi$  with the latitude  $\varphi$  of the lower corner of the triangle. Thus  $w$  is a measure of the latitude of the lower vertices of the triangle in Figure 3.

Two adjacent main spherical triangles are combined to form a “diamond”, i.e. a logical square block. Five of the diamonds originate from the north pole and five from the south pole. The numbering and order of the diamonds are outlined in Figure 4.

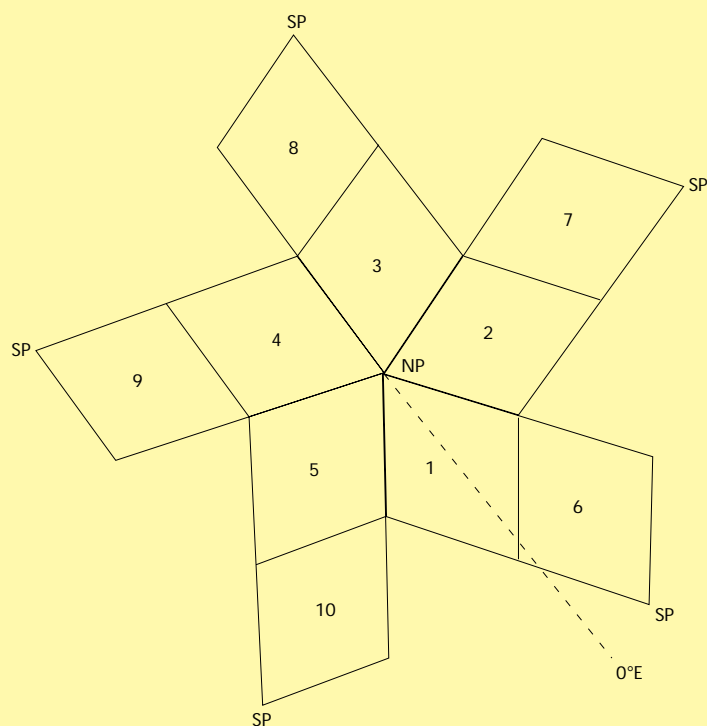


Figure 4. The 20 main spherical triangles combined to 10 diamonds.

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Diamonds 1 to 5 are the “northern” ones, i.e. they start at the north pole, while diamonds 6 to 10 start at the south pole. The so-called home vertex of each diamond (in the order 1, 6, 2, 7, 3, 8, 4, 9, 5, 10) is shifted by  $\pi/5$  to the east starting at  $-\pi/5$  for the first diamond. Thus the 10 home vertices have the geographical coordinates ( $\lambda$  and  $\varphi$ ) on the unit-sphere as presented in Table 1.

Table 1  
Geographical coordinates ( $\lambda$  and  $\varphi$ ) of the home vertices of the 10 diamonds

Diamond #	1	2	3	4	5	6	7	8	9	10
$\lambda$	$-\pi/5$	$\pi/5$	$3\pi/5$	$5\pi/5$	$-3\pi/5$	0	$2\pi/5$	$4\pi/5$	$-4\pi/5$	$-2\pi/5$
$\varphi$	$\pi/2-w$	$\pi/2-w$	$\pi/2-w$	$\pi/2-w$	$\pi/2-w$	$w-\pi/2$	$w-\pi/2$	$w-\pi/2$	$w-\pi/2$	$w-\pi/2$

A Cartesian coordinate system is placed into the unit-sphere with the origin in the centre of the sphere, the z-axis towards the north pole and the x-axis in the direction of the Greenwich meridian. The Cartesian coordinates ( $x, y, z$ ) of a point on the unit-sphere follow from equation (2):

$$\begin{aligned} x &= \cos \lambda \cos \varphi = \cos \lambda \sin w \\ y &= \sin \lambda \cos \varphi = \sin \lambda \sin w \\ z &= \sin \varphi = \cos w \end{aligned} \quad (2)$$

Thus the two pole vertices have the Cartesian coordinates (0, 0, 1) and (0, 0, -1), respectively.

The geographical coordinates ( $\lambda, \varphi$ ) of a point on the unit-sphere with the Cartesian coordinates ( $x, y, z$ ) follow from equation (3) which may be derived from equation (2):

$$\begin{aligned} \lambda &= \arctan \frac{y}{x} \\ \varphi &= \arcsin z \end{aligned} \quad (3)$$

For the grid generation, the sides ( $w$ ) of the 20 main triangles are iteratively subdivided into  $n_i$  equal parts to form sub-triangles. Each point in a main triangle is now surrounded by six triangles (Figure 5) and is, therefore, in the centre of a hexagon (see also Figure 6). However, the points which form the vertices of the icosahedron

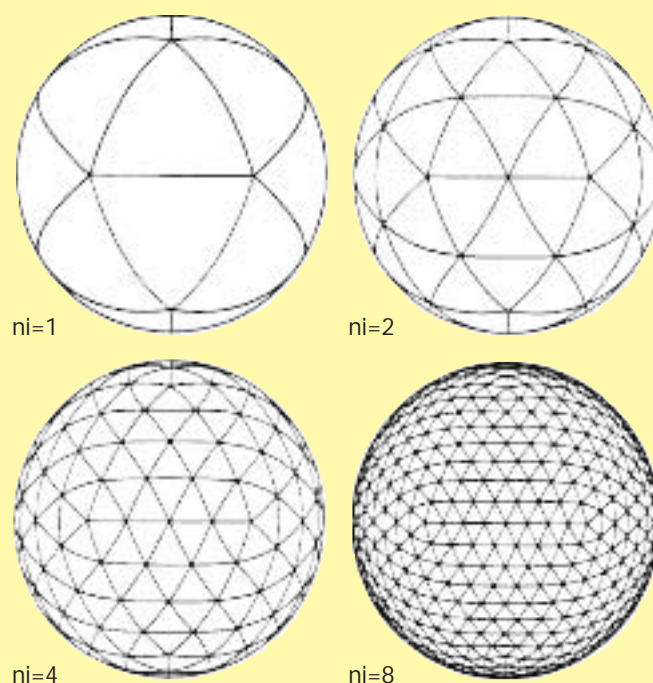


Figure 5. Spherical triangular grids for different values  $n_i$  of the subdivision of the main spherical triangles.

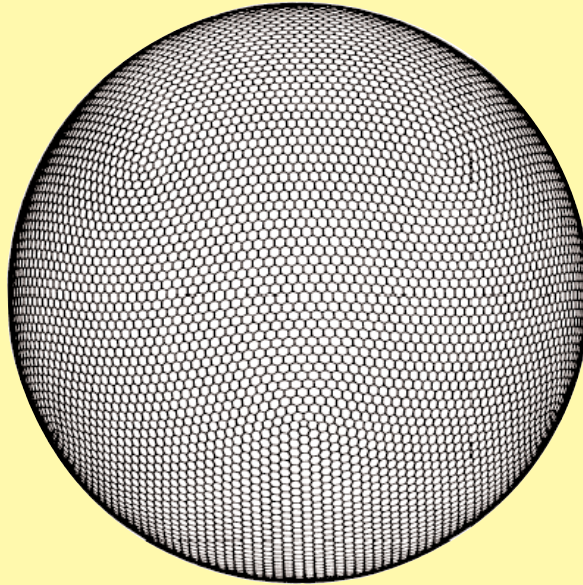


Figure 6. Polygons which represent the area of representativeness of a triangular grid-point.

are surrounded by only five triangles and therefore these 12 special points are the centres of pentagons. For the first subdivision,  $w$  may be divided into three parts, later on, only bisections are allowed. This restriction is due to the use of a multi-grid (MG) solver for the Helmholtz equations in the semi-implicit time stepping. MG solvers work efficiently with such mesh refinements. Thus the number ( $n_i$ ) of subdivisions of  $w$  is factorized according to equation (4):

$$n_i = 3^{n_3} 2^{n_2} \quad (4)$$

with  $n_3 = 0$  or  $1$  and  $n_2 \geq 0$ . Figure 5 shows the resulting grids for  $n_i = 1, 2, 4$  and  $8$ , i.e.  $n_2 = 0, 1, 2, 3$  with  $n_3 = 0$ .

The model grid-points (nodes) are located at the vertices of the triangles; thus there are  $(n_i+1)^2$  grid-points within one diamond. Of these  $(n_i+1)^2$  grid-points,  $n_i \times n_i$  are "uniquely" identified with each diamond; one extra row and column is shared between neighbouring diamonds.

On Earth with a mean radius  $R_E = 637\,122.9$  m, the length ( $L$ ) of a side of a main triangle is  $L = wR_E = 705\,389.8$  m. The mesh size ( $\Delta$ ) of the triangular grid with  $n_i$  equal intervals on the side of a main triangle is not constant within a diamond but varies by 20 per cent at most on the sphere and is approximately given by using equation (5). For example, for  $n_i = 32$ ,  $\Delta$  varies between 220 and 263 km, for  $n_i = 64$ ,  $\Delta$  varies between 110 and 132 km and for  $n_i = 128$ ,  $\Delta$  varies between 55 and 66 km:

$$\Delta \approx \frac{wR_E}{n_i} \quad (5)$$

The number  $N$  of grid-points, not counting the common edges of the diamond, is given by equation (6):

$$N = 10 n_i^2 + 2 \quad (6)$$

Table 2a gives the mesh size, the number ( $N$ ) of grid-points and the time step ( $\Delta t$ ) for different values of  $n_i$ , if only bisections are performed, i.e.  $n_i = 2^{n_2}$ . The time step ( $\Delta t$ ) is calculated under the assumption that an air parcel does not leave the region of the six surrounding triangles during the period of twice the time step, i.e.  $2 \Delta t < h/v_{Max}$ , with the height ( $h$ ) of the spherical triangle (which is the shortest distance for leaving a triangle) and  $v_{Max}$ , the maximum wind speed ( $\approx 125$  m s<sup>-1</sup>) assuming that the fast gravity waves are treated semi-implicitly. The height ( $h$ ) of a spherical triangle approximately follows from equation (7) and is about 5 per cent smaller than the mesh size ( $\Delta$ ):

$$h \approx \arcsin \left( \sin \frac{w}{n_i} \sin \frac{2\pi}{5} \right) R_E \quad (7)$$

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Table 2a  
Mesh size ( $\Delta$ ), height ( $h$ ), number ( $N$ ) of grid-points and time step ( $\Delta t$ ) for the spherical triangular mesh using only bisections

$n_i$	16	32	64	128	256
$\Delta$ (km)	441	220	110	55	28
$h$ (km)	420	210	105	52	26
$N$	2 562	10 242	40 962	163 842	655 362
$\Delta t$ (s)	1 600	800	400	200	100

Table 2b  
Mesh size ( $\Delta$ ), height ( $h$ ), number ( $N$ ) of grid-points and time step ( $\Delta t$ ) for the spherical triangular mesh using first a trisection followed by bisections

$n_i$	12	24	48	96	192
$\Delta$ (km)	588	294	147	73	37
$h$ (km)	559	279	140	69	35
$N$	1 442	5 762	23 042	92 162	368 642
$\Delta t$ (s)	2 200	1 100	550	275	138

Each grid-point is representative for a spherical polygon with six vertices (Figure 6) except the 12 vertices of the icosahedron which are surrounded by five triangles only. The grid-point indices are defined as presented in Figure 7.

The start address (0, 1) reflects the philosophy that the  $n_i \times n_i$  grid-points which are "uniquely" identified within each diamond have the indices 1 to  $n_i$  for rows and columns. The extra row and column needed for communication between neighbouring diamonds is lying in one case at the beginning of the first coordinate and in the other case at the end of the second. Thus points outside the range (1: $n_i$ , 1: $n_i$ ) belong to the neighbouring diamonds and have to be communicated during each time step. Grid-point (0, 1), respectively, is the north pole for diamonds 1 to 5, and the south pole for diamonds 6 to 10.

The calculation of the subdivision of the great circle between two points  $P_1$  (with location vector  $x_1$ ) and  $P_2$  (with location vector  $x_2$ ) can be derived from Figure 8.

Since  $x_1$  and  $x_2$  define the great circle plane through  $P_1$  and  $P_2$ , all points ( $P$ ) with the location vector ( $x$ ) on the great circle may be written as a linear combination of  $x_1$  and  $x_2$ :

$$x = \alpha x_1 + \beta x_2 \quad (8)$$

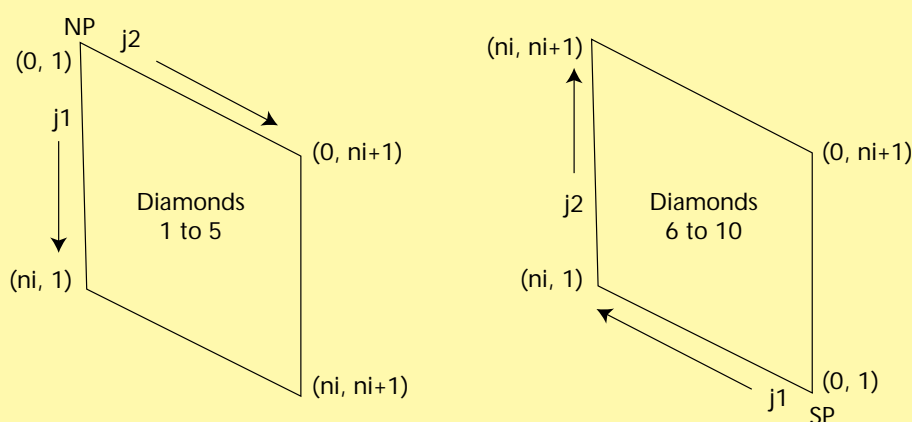


Figure 7. Grid-point indices for a northern (left) and southern (right) diamond.

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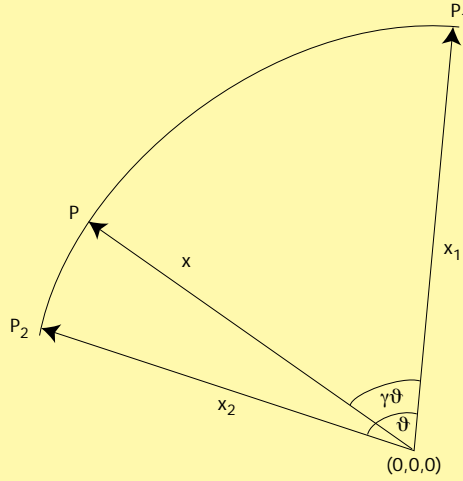


Figure 8. Calculation of the subdivision of the great circle through the points  $P_1$  and  $P_2$  on the unit-sphere.

The coefficients  $\alpha$  and  $\beta$  are derived from the condition that  $x$  is a vector on the unit-sphere and the angle between  $x$  and  $x_1$  is given by  $\gamma\theta$  with  $\gamma$  between 0 and 1 and  $\theta$  being the angle between  $x_1$  and  $x_2$ , i.e. the length of the great circle between  $P_1$  and  $P_2$ :

$$\begin{aligned} x \cdot x &= 1 = \alpha^2 + \beta^2 + 2\alpha\beta \cos \theta \\ x \cdot x_1 &= \cos(\gamma\theta) = \alpha + \beta \cos \theta \end{aligned} \quad (9)$$

Substituting  $\alpha$  from the second equation into the first one, the coefficients follow from equation (10):

$$\begin{aligned} \alpha &= \frac{\sin((1 \pm \gamma)\theta)}{\sin \theta} \\ \beta &= \frac{\sin(\gamma\theta)}{\sin \theta} \end{aligned} \quad (10)$$

The angle  $\theta$  between  $x_1$  and  $x_2$  follows from the scalar product  $x_1 \cdot x_2$  or by calculating the distance ( $d$ ) between  $x_1$  and  $x_2$  and observing that  $\sin \theta/2 = d/2$ .

The grid-point coordinates ( $x, y, z$ ) of all triangle vertices on the unit-sphere are derived from equation (8) using the coefficients of equation (10). The  $(n_i+1)^2$  grid-points in a diamond form the vertices of  $2n_i^2$  triangles (Figure 9) and half of those point northward and half southward.

To calculate the coordinates ( $x_c, y_c, z_c$ ) of the triangle centres  $P_c$ , the coordinates of the three triangle vertices  $P_1, P_2$  and  $P_3$  are summed and normalized as in equation (11):

$$\begin{aligned} x_c &= (x_1 + x_2 + x_3) \cdot x_N \\ y_c &= (y_1 + y_2 + y_3) \cdot x_N \\ z_c &= (z_1 + z_2 + z_3) \cdot x_N \end{aligned} \quad (11)$$

with

$$x_N = \frac{1}{\sqrt{(x_1 + x_2 + x_3)^2 + (y_1 + y_2 + y_3)^2 + (z_1 + z_2 + z_3)^2}}$$

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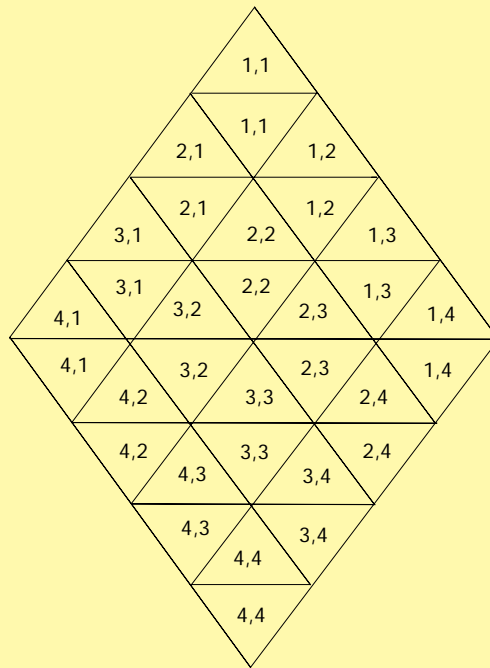


Figure 9. The  $2n^2$  triangles in a diamond defined by the  $(n+1)^2$  vertices for  $n = 4$ .

The area of the  $2n^2$  triangles in a diamond can be calculated by using equation (12) which is due to Huilier. The triangle sides are denoted by  $a$ ,  $b$  and  $c$ . On the unit-sphere, the excess angle is equal to the area of the spherical triangle:

$$\tan \frac{\varepsilon}{4} = \sqrt{\tan \frac{s}{2} \tan \frac{s-a}{2} \tan \frac{s-b}{2} \tan \frac{s-c}{2}} \quad (12)$$

with:

$$s = \frac{1}{2} (a + b + c)$$

Since each grid-point is surrounded by six triangles (five triangles at the 12 special points), the grid-point is the centre of a hexagon (pentagon at the 12 special points) as is illustrated in Figure 10. The coordinates of

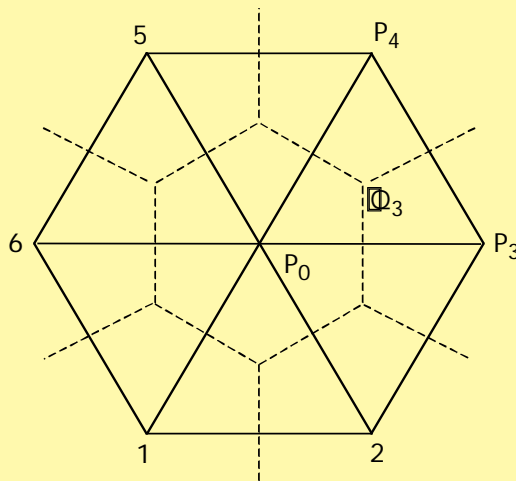


Figure 10. Hexagon connected to a grid-point of the triangular mesh.

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the vertices of the hexagon, i.e. points  $Q_1, Q_2, \dots, Q_6$ , are in a good approximation given by averaging the Cartesian coordinates of the three surrounding triangles vertices and normalizing to unit length, thus they follow from equation (11).

The grid-point in the centre of the hexagon is denoted by 0, the six surrounding triangles (and their vertices) by 1 to 6 counting counter-clockwise. We define point  $Q_i$ , i.e. a vertex of the hexagon, equidistant from the three vertices  $P_0, P_i$ , and  $P_{i+1}$  such that  $Q_i$  and  $Q_{i+1}$  is the perpendicular bisection of the great circle  $P_0P_{i+1}$  (Figure 10). The coordinates of  $Q_i$  are needed for the calculation of the topographical fields like orography, land fraction, roughness length as mean values over the area of the hexagons. Here, high-resolution datasets are averaged over the hexagon area.

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